Dokl. Akad. Nauk SSSR
Tom 173 (1967), No. 3

Soviet Math. Dokl.
Vol. 8 (1967), No. 2

UNIFORM ASYMPTOTICS OF LINEAR SYSTEMS

A. M. MOLČANOV

We propose a new way of studying asymptotics of systems of linear equations with a small parameter multiplying the derivative

$$\varepsilon \frac{d\tilde{x}}{dt} = A(t, \varepsilon)\tilde{x}. \tag{1}$$

A fundamental matrix of the systems is represented in the form

$$S(t, \varepsilon) = W(t, \varepsilon)\Lambda(t, \varepsilon), \tag{2}$$

where the matrix Λ is diagonal and the matrix $W(t, \epsilon)$ has a limit as $\epsilon \to 0$.

The determination of the matrix W can be reduced to the calculation of the factors of an infinite (to the right) product

$$W = W_0 W_1 \cdots W_n \cdots, \tag{3}$$

where each factor W_n can be found by a quadrature, and where the product converges more rapidly than the numerical product

$$\prod_{n=0}^{\infty} (1 + C\varepsilon^{\Phi_n}). \tag{4}$$

Here the Φ_n are the Fibonacci numbers, which grow like $((\sqrt{5} + 1)/2)^n$. In the special case of a second-order matrix the rate of convergence is still more rapid and reaches the rate of convergence of Newton's method.

The assertions of this paper are proved for an arbitrary interval $t_0 \le t \le t_1$ on which the real parts of the eigenvalues of the matrix $A(t, \epsilon)$ are all distinct.

1. Matrix A triangular. We also assume that the diagonal elements of the matrix A, which in this case are the eigenvalues, are arranged in order of decreasing real parts

$$\operatorname{Re} \lambda_1(t, \varepsilon) > \operatorname{Re} \lambda_2(t, \varepsilon) > \ldots > \operatorname{Re} \lambda_k(t, \varepsilon). \tag{5}$$

In this important special case the matrix W is also triangular, and can be determined by quadratures.

Let us write the equation satisfied by the matrix $S(t, \epsilon)$:

$$\varepsilon\, dS / dt = AS, \tag{6}$$

and then substitute (2) into (6), multiplying on the right by Λ^{-1} and on the left by W^{-1}. We obtain

$$\varepsilon W^{-1}\dot{W} + \varepsilon\dot{\Lambda}\Lambda^{-1} = W^{-1}AW. \tag{7}$$

We seek a triangular matrix W with 1's on the diagonal. The left member of (7) then decomposes into two "nonintersecting" matrices. The second term has nonzero elements only on the diagonal,

while the first has nonzero terms only below the diagonal. It is not difficult to show that in this case the single equation (7) uniquely determines two equations for W and Λ. However, we shall introduce certain notation for future convenience. If $z = \|z_{il}\|$ is an arbitrary matrix, we denote by $T(z)$, $\lambda(z)$, and $\perp(z)$ the lower triangular, diagonal, and upper triangular parts of z respectively:

$$T(z) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ z_{21} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ z_{n1} & z_{n2} & \cdots & 0 \end{pmatrix}, \quad \lambda(z) = \begin{pmatrix} z_{11} & 0 & \cdots & 0 \\ 0 & z_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & z_{nn} \end{pmatrix},$$

$$\perp(z) = \begin{pmatrix} 0 & z_{12} & \cdots & z_{1n} \\ 0 & 0 & \cdots & z_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{8}$$

We now write the equations for Λ and W:

$$\varepsilon\dot{\Lambda} = \lambda\Lambda, \tag{9}$$

$$\varepsilon\dot{W} = TW + \lambda W - W\lambda, \tag{10}$$

where $\lambda = \lambda(A)$ and $T = T(A)$.

Equation (9) can be integrated directly:

$$\Lambda(t, \varepsilon) = \exp\left[\frac{1}{\varepsilon}\int_{t_0}^{t} \lambda(\tau, \varepsilon)\, d\tau\right]. \tag{11}$$

Using the intermediate substitution $W = \Lambda U \Lambda^{-1}$, equation (10) can be written as the integral equation

$$W(t) = E + \frac{1}{\varepsilon}\int_{t_0}^{t} \exp\left[\frac{1}{\varepsilon}\int_{\tau}^{t} \lambda(s)\, ds\right] T(\tau)\, W(\tau) \exp\left[-\frac{1}{\varepsilon}\int_{\tau}^{t} \lambda(s)\, ds\right] d\tau. \tag{12}$$

If this equation is integrated k times, an exact equation is obtained, since the product of more than k triangular matrices with zeros on the diagonal (k is the order of the matrices) is identically zero. Moreover it can be directly checked that all exponents which are under the integral sign will have negative real part. This follows (after multiplying out the exponential factors of course) also from the triangular nature of the matrices T and W and from the ordering (5) of the eigenvalues of the matrix A. We may therefore take $t^* = t_0$, and the matrix $W(t, \epsilon)$ determined by equation (12) will have a finite limit as $\epsilon \longrightarrow 0$. This limit is a solution of the equation

$$TW + \lambda W - W\lambda = 0, \tag{13}$$

which is obtained from (10) by formally setting $\epsilon = 0$. Equation (13) has a unique solution W among triangular matrices with 1's on the diagonal. From (12) we can deduce the estimate

$$W = E + O(T)$$

for small T, which will play an important role in the future. In the estimate there appear in the denominator the differences of the real parts of the eigenvalues of the matrix A. Hence condition (5) is essential for the validity of the estimate.

Remark. We can examine quite analogously the case where the matrix A has zeros below the diagonal. We need only replace $T(A)$ by $\perp(A)$ everywhere, and set $t^* = t_1$ in (12).

2. Matrix A almost triangular. We now study a somewhat more general case, where the matrix A

is not precisely triangular, but where its departure from triangularity is slight. For example let

$$\perp(A) \sim \delta,$$

where δ is a small quantity.

We again seek a solution of the equation for a fundamental matrix of the system in the form of a matrix product

$$S = P\Sigma, \quad \varepsilon P^{-1}\dot{P} + \varepsilon\dot{\Sigma}\Sigma^{-1} = P^{-1}AP.$$

There is some arbitrariness about P in this equation, and we make use of this by setting

$$\varepsilon\dot{P} = \mathrm{T}P + \lambda P - P\lambda.$$

This equation for P is written as if the matrix A were triangular. If it were in fact triangular, then the equation for Σ would give a diagonal matrix. But since the matrix A is only almost triangular, we expect to obtain for Σ an almost diagonal equation. After this it is to be hoped that the situation can be further improved by repeating the procedure. When we write the equation for Σ we have

$$\epsilon\dot{\Sigma} = B\Sigma,$$

where $B = P^{-1}\perp(A)P + \lambda(A)$.

Our expectations are in fact justified, since now the matrices B turn out to be of order δ not only "above" but also "below". In addition, we see that the equation for Σ has the same form as the original equation for S, but with a different matrix B. The matrix B is better than A since, although the upper triangular part is not decreased, the lower part is decreased. It is therefore natural that at the following step the upper triangular part should be improved.

These considerations suggest the appropriateness of the following iterative process. Starting with a matrix A_n construct in sequence the four matrices P_n, B_n, Q_n, A_{n+1}:

P_n is the solution of the equation

$$\varepsilon\dot{P}_n = \mathrm{T}(A_n)P_n + \lambda(A_n)P_n - P_n\lambda(A_n); \tag{14}$$

B_n is given by

$$B_n = P_n^{-1}\perp(A_n)P_n + \lambda(A_n); \tag{15}$$

Q_n is then the solution of the equation

$$\varepsilon\dot{Q}_n = \perp(B_n)Q_n + \lambda(B_n)Q_n - Q_n\lambda(B_n) \tag{16}$$

and finally A_{n+1} is given by

$$A_{n+1} = Q_n^{-1}\mathrm{T}(B_n)Q_n + \lambda(B_n). \tag{17}$$

This method is similar to the alternating method in conformal mapping. Here we decrease first one and then the other half of the matrix. Note that the diagonal elements always remain of the order of unity and coincide in their principal terms with the initial terms. Let us see what is achieved as the result of one such double step. An estimate is obtained by use of the formula

$$P_n = E + O[T(A_n)].$$

Substituting into (15) and retaining only principal terms we find that $B_n \approx \lambda(A_n) + \perp(A_n) + O(\mathrm{T}(A_n)\perp(A_n))$. Hence $\perp(B_n) \sim \perp(A_n)$, $\mathrm{T}(B_n) \sim \perp(A_n)\mathrm{T}(A_n)$.

In exactly the same way, at the following halfstep we obtain $\perp(A_{n+1}) \sim \perp(B_n)\,\mathrm{T}(B_n)$, $\mathrm{T}(A_{n+1}) \sim \mathrm{T}(B_n)$.

These relations guarantee sufficiently rapid convergence, although not so rapid as with Newton's method, since the error is squared not at every stage but only at the first stage. It is easy to find the rate of decrease of the nondiagonal terms. Let

$$\mathrm{T}(A_n) \sim \delta^{p_n}, \quad \perp(A_n) \sim \delta^{q_n};$$

we then find from the above formulas

$$\mathrm{T}(B_n) \sim \delta^{p_n+q_n}, \quad \perp(B_n) \sim \delta^{q_n}$$

and for the second halfstep we obtain

$$\mathrm{T}(A_{n+1}) \sim \delta^{p_n+q_n}, \quad \perp(A_{n+1}) \sim \delta^{p_n+2q_n}.$$

We can therefore write

$$p_{n+1} = p_n + q_n, \quad q_{n+1} = p_n + 2q_n;$$

since for $n = 0$ we had $p_0 = 0$, $q_0 = 1$, the resulting sequence which is easily seen to give an upper bound coincides with the Fibonacci numbers. More accurate calculations show that for matrices of second order the rate of convergence obtained is precisely that of Newton's method. Curiously the pair is the first convergent of the continued fraction for $(\sqrt{5} + 1)/2$, the asymptotic ratio of the Fibonacci numbers. It is possible that a more accurate calculation for third-order matrices would give a rate corresponding to the third convergent (we must approximate in excess), for fourth-order matrices a rate corresponding to the fifth convergent, etc. However, we shall not derive more detailed estimates.

3. $A(t, \epsilon)$ an arbitrary differentiable matrix. In this case the problem is easily reduced to that of the preceding section. It is only necessary to make a nonstandard first step in the iterative process.

Let us in fact consider the original equation for the matrix S, and as usual set $S = W_0\Sigma$. For Σ, as we have already seen three times, there is obtained an equation of the same form as for S, but with the matrix B:

$$B = W_0^{-1}AW_0 - \varepsilon W_0^{-1}\dot{W}_0.$$

We now choose the matrix W_0 so that the first, principal, term in the expression for B is triangular. There always exists such a matrix to accomplish this, and it can even be chosen to be unitary. It is very simple to construct. Take a system of eigenvectors of the matrix A and orthogonalize them. Then for W_0 we may use the matrix for transformation of coordinates to the orthonormal basis thus obtained. There are exactly $k!$ ways of choosing such a matrix W_0. However, for our purposes, only two are suited. One of these arranges the eigenvalues in order of increasing real parts, while the other arranges them in order of decreasing real parts.

It should be noted that the unitary matrix $W_0(t, \epsilon)$ thus obtained will have bounded derivatives only on an interval on which the eigenvalues remain distinct. At points at which the matrix $A(t)$ has multiple roots, the derivative $\dot{W}_0$ will become infinite as a rule. This is easily verified for the case of a turning point of the Schrödinger equation, for example.

Received 21/APR/66

Translated by W. S. Loud